

Dynamical systems approach to the Busy Beaver problem

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Abstract

The purpose of this paper is to reconsider the Busy Beaver problem, which is a modification of the Turing machine's halting problem, in the light of a dynamical systems approach. Numerical experiments show that the spatio-temporal patterns of machines can be roughly classified into four types. The spatio-temporal patterns of the random become more regulated when machines move longer before they halt. In particular, well-regulated and self-similar patterns are found in most longer-lived machines, which include known best Busy Beaver machines. A relation to the Collatz system is briefly discussed.

Key words: Turing machine halting problem, dynamical system, Collatz $3k + 1$ mapping, spatio-temporal pattern

1. Introduction

The Turing machine halting problem is a central issue of computability theory, and many modifications are known. Among them, the Busy Beaver problem is famous for its easy definition, and is described as follows.

The Turing machine has finite internal states, a tape, and a transition table. The machine reads a symbol from the tape where the machine's head is pointing, rewrites a symbol according to the transition table, transits its own state, and moves the head right or left. This process continues until the machine reaches a halting state. When starting from an all-blank tape, some machines even-

tually halt, or never halt. The Busy Beaver is a machine with a given state size that halts with the maximum number of non-blank symbols left on the tape. Alternatively, the machine halts with the maximum number of time steps before halting. When one makes a function of the beaver's score against its state size, this function grows faster than any computable function, and therefore turns out uncomputable [8].

Since this problem was first stated by T. Rado in [8], many computational studies have been performed to search for a small state beaver. Less than 5-state beavers are determined by computer assisted proofs, and lots of candidates with 5 or more states are discovered [4] [2] [5]. For example, Marxen et al. found more than 5×10^{43} time steps (and finally halt!) with 6-state machines in 2000 [5]. Algorithms of some 5-6-state record holders are analyzed to be equivalent to the generalized

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Collatz $3k + 1$ mapping [6] [7].

In this paper, we focus on the dynamics of machine behavior on the tape, not restricted to high-scored machines. From a dynamical system’s point of view, low-scored machines, which are in the absolute majority, give relatively random motions compared to well-regulated high-scored record holders. The halting probability distributions show that there is a smooth transition from an ad hoc random halting machine to a regular Collatz-like system. Also, we show spatio-temporal patterns of machines, all patterns are roughly divided into 4 types, and show that the best Busy Beaver so far can be found in type I.

2. Statistical Analysis

2.1. Halting Probability

Some numerical approaches have been made to characterize uncomputable problems, such as the maximal time steps before machines halt, the maximal spaces where machine spends, the maximal numbers of ones machine leaves, and so on[1]. Rado first theoretically studied the halting probability of randomly generated Turing machines as a function of the number of given machine states. This function is proved to be uncomputable by himself [8]. We begin by studying the Rado’s halting probability of the Busy Beaver problem numerically.

2.2. Method and Results

We concern here finite n -state Turing machines, and a tape with $\{0, 1\}$ symbols. The halt state is not included in n . In order to study the Busy Beaver problem, we simulate the set of randomly generated machines with the initial blank tape. That is, Rado’s function is studied here with machines with the blank input tape. For more than 5 state machines, the transition table is generated by the Monte Carlo method. Otherwise, we elaborate all possible tables.

The simulation results for $n = 5, 10, 20$ are shown in Fig.1, which denote the probability densities of machines that halt at each time step. By

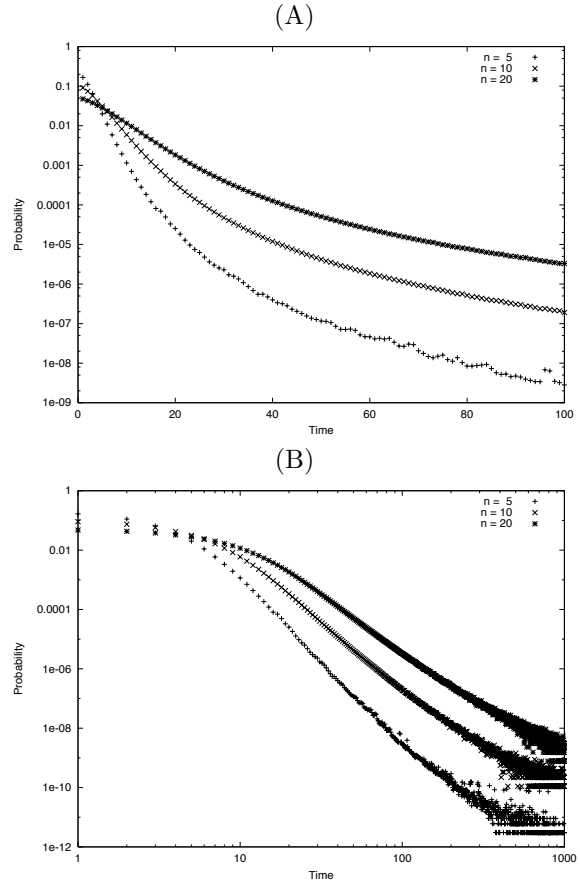


Fig. 1. Probability density for $n = 5, 10, 20$. (A) is on log scale, (B) is on log-log scale.

integrating them, we calculate the halting rate of machines as a function of the internal states. We should note that Fig.1 shows only a part, as infinite time integration is needed to take an accurate measurement of halting probability, hence, uncomputable.

As Fig.1 indicates that probabilities fall exponential in the early stage, gradually switch to the power, and the index of power also changes. This power-law behavior is also reported in [9] Let n be the state size, t be time, $P_n(t)$ be a density function. We approximately estimate the function as,

$$P_n(t) = A \exp(-at),$$

$$a \propto n^{-0.845} \quad (1)$$

Also, in the power law region,

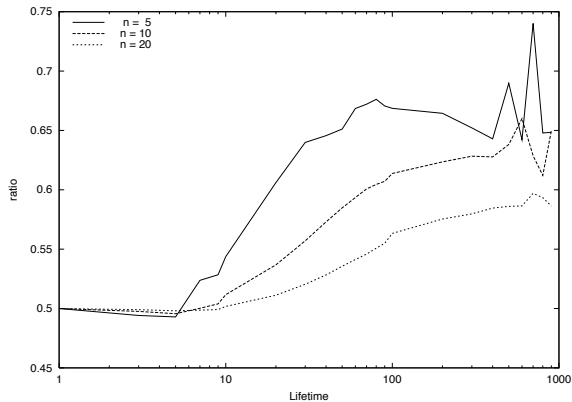


Fig. 2. Average ratio of writing ones on the tape for $n = 5, 10, 20$. Lifetime axis is on log scale.

$$P_n(t) = Bt^{-bt},$$

$$b \propto n^{-0.236} \quad (2)$$

We have not yet obtained an analytic solution of these results, however, we interpret that machines halt at random in the early stage, and eventually halt in later stages by different causes. This power law behavior is now related with the machine's spatio-temporal behaviors.

2.3. Order parameter of machines' behaviors

We propose an order parameter to characterize machine's motions on a tape, by computing a frequency of writing **0/1** symbols on a tape before a machine halts. If a machine writes symbols at random, the average rate of writing a symbol **1** will become 0.5. The deviation from 0.5 indicates that the corresponding machine behavior is more structured.

Fig.2 shows the simulation results for $n = 5, 10, 20$. The ratio looks steady around 0.5 for the early stage of machines' motion. Beyond a certain period of time, the rates deviate from 0.5. Machines with more states tend to deviate more from 0.5. It should be noted here that the time when the machines begin to show non-random behaviors coincides with the time when the probability density drops off from the exponential form. Beyond the point, the machine behaviors become more structured. The non-random structure is revealed

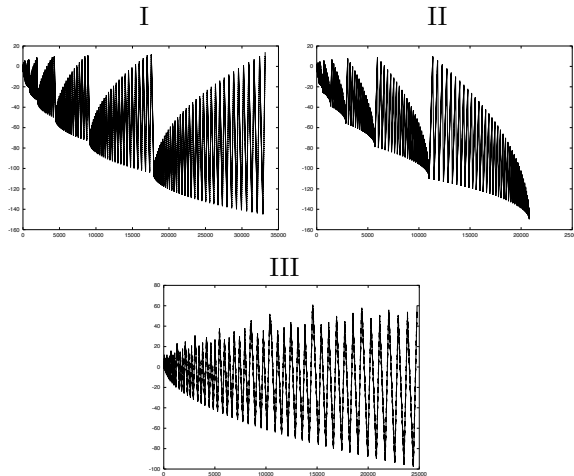


Fig. 3. Majority spatio-temporal pattern of 5-state high-score records. We take the horizontal axis as time, and the vertical axis as tape position. Types I and II show self-similar patterns. Type III also have repetitive motion but close observation shows that it is relatively chaotic.

on the spatio-temporal pattern on a tape in the next section.

3. Spatio-temporal patterns

Behaviors of machines with a lifetime greater than 1000 are outstanding and very different from each other. To characterize them in this region, we classify the spatio-temporal patterns on the tape. We limit our discussion to the case of 5-state machines, but some may be applied to 6-state or higher machines. There are roughly 3 regular types and an unclassified type. Fig.3 shows majority patterns and Fig.4 shows some examples of the unclassified types. Compared with the majority patterns, unclassified types are some random modulations of self-similar ones.

For life-times longer than 10^4 , any machine behavior falls into one of the three types (see Fig.3). Especially, types I and II have well-regulated self-similar features, and the Busy Beavers and record holders are found in type I. The number of unclassified behavior decreases for longer lifetime machines (Fig.5).

It is shown that 5- or 6-state record holder

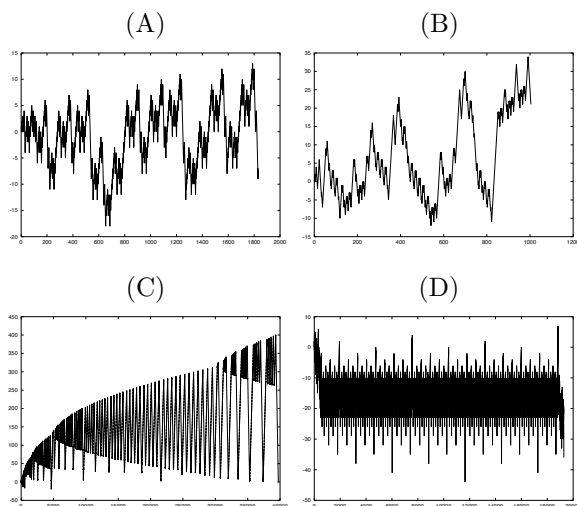


Fig. 4. Some examples of the unclassified types. (A) & (B) show partially recursive but relatively random motion. (C) is complex type of I & III. (D) shows counter-like motion.

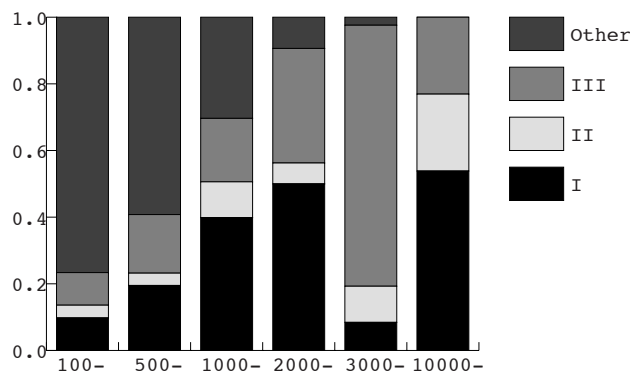


Fig. 5. Population of the spatio-temporal pattern types of 5-state machines. Others had partially repetitive motions.

machines are imitated by a generalized Collatz mapping [6] [7]. The Collatz mapping is an iterated function on integers, consisting of a modulo test and some linear functions [7]. In comparison with Fig.3, the linear function determines the self-similar units of the patterns, and the modulo test is executed at the boundary of those units.

4. Discussion

We argue that self-similarity in spatio-temporal patterns of long-lived machines may provide the power-law behavior in the probability density of halting states. A requirement for Busy Beaver machines is to write many 1s but to halt at a certain moment. In this sense, a simple one-way moving behavior is removed (if they move n steps before halting has simply n states.). Longer-lived machines with a few states (e.g. 5) have to apply the same rule set recursively to generate future input patterns. The resulting spatio-temporal pattern has successive turning points (t_1, t_2, \dots, t_n) before halting. The time duration between successive turning points increases sometimes polynomial in n but sometimes shows more complex behavior. We observe that the machines with the power-law behavior in the turning points tend to have the power-law behavior in their halting probability density function. The details will be reported elsewhere.

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