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Emergence of a New Attracting Set by a Mixed Strategy in Game Dynamics

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Game dynamics with a mixed strategy player is studied. A new attracting set emerges under a certain condition by introducing a mixed strategy into a system with a pure strategies system. In terms of a replicator system, introducing a mixed strategy means to increase the number of rows and columns of the interaction matrix by 1 without changing its rank size. As a result, a system has a neutral dimension and a new attracting set emerges on the center manifold of a fixed point. A possible scenario of evolution via a mixed strategy is briefly discussed.

KEYWORDS: replicator dynamics, evolutinary game theory, mixed strategy

A game theory defines a finite n-person normal form game as a set of n pure strategies with an associated $n \times n$ payoff matrix. If strategy i is played against strategy j, the expected payoff to strategy i is simply given by A_{ij}. Any mixed strategy assigns probabilities to each pure strategy.

Maynard Smith had introduced a notion of evolutionary game dynamics[1] in which he considers the population dynamics of players using pure/mixed strategy. The simplest form of this dynamics was developed by Maynard Smith[2,3] and Taylor and Jonker,[4] and is now widely studied as the *replicator dynamics system*.

The time evolution of replicator dynamics having n pure strategies with the pay-off matrix A is given by

$$\dot{x}_i = x_i \left(\sum_j A_{ij} x_j - \sum_{j,k} x_j A_{jk} x_k \right), \tag{1}$$

where x_i denotes the relative frequency of the player using pure strategy i. Hence the conditions $\sum x_i = 1$ and $x_i \geq 0$ should be satisfied.

Some new and interesting behaviors of this dynamics as a dynamical system have been found.[5–8] In this letter, we study a role of mixed strategy in the population of pure strategies.

Previously, in the book of evolutionary game theory by Maynard Smith,[3] it has been argued that game behaviors change depending on whether each player can use a pure or a mixed strategy. An evolutionarily stable strategy (ESS) is used for the solution concept of a game system; when most individuals adopt the ESS strategy, no mutant strategy can invade the population. However, whether or not an ESS with only pure strategies ensures a stable polymorphic population using mixed strategies is not fully understood.[9–11] As is discussed by Zeeman,[9] we should study the dynamics of a probability density over a strategy space. While this density dynamics has not been fully analyzed, some phenomenological studies have been reported. Examples can be found in some three-strategy games[3,10] and more recently in the prisoner's game with one memory strategy.[12] In the former case, ESS disappears and in the latter case a new ESS (as an equilibrium point) appears at a boundary. In this letter, we report how a new attracting set, not restricted to an equilibrium ESS, can emerge by having mixed strategies.

In addition to the ESS arguments, mixed strategies are important to study since they open up a new way of understanding evolution. The original replicator system has a fixed state space. Study of evolution requires changing the state space or the updating of the interaction matrix; however, we have no realistic way of doing so (cf. Tokita and Yasutomi[13]). Moreover, we do not believe that evolution always creates a new character. As Ohno[14] and Jacob[15] argue, the evolutionary process can be taken as a bricolage; it does not produce novelties from scratch but from recombinations of old materials. We return to this point later, and proceed to the analysis of replicator game dynamics with a mixed strategy.

An extension of game dynamics to include mixed strategies is straightforward. Here we assume that the number of pure strategies (n) is finite and that of mixed strategies (m) is also finite. The probability which the mixed strategy k assigns to the pure strategy i is denoted by $\boldsymbol{s}_i^k$. Hence the expected payoff associated with the mixed strategy l is $\sum_{i,j} \boldsymbol{s}_i^k A_{ij} \boldsymbol{s}_j^l$, where $\boldsymbol{s}^k$ and $\boldsymbol{s}^l$ are n-component vectors with the conditions $s_i^k \geq 0$, $s_i^l \geq 0$ and $\sum_{i=1}^n s_i^k = \sum_{i=1}^n s_i^l = 1$. Using n-column vectors $\boldsymbol{s}^i$ ($i = 1, \ldots, m$) on a pure strategy space, we rewrite the game dynamics as

$$\dot{y}_i = y_i \left(\sum_j {}^t\boldsymbol{s}^i A \boldsymbol{s}^j y_j - \sum_{j,k} y_j {}^t\boldsymbol{s}^j A \boldsymbol{s}^k y_k \right), \tag{2}$$

where y_i denotes the relative frequency of a player using strategy i. A pure strategy is defined as a vector whose components have $n-1$ zeros and a single unity element.

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Comparing eq. (2) with eq. (1), we notice that this equation has a new interaction matrix B_{ij} $(= {}^t\boldsymbol{s}^i A s^j)$ and is defined on the $m-1$ dimensional simplex instead of on $n-1$. Namely, the relative frequency of pure strategy x_i is now given by $\sum_j \boldsymbol{s}_i^j y_j$. Defining the $n \times m$ matrix $\boldsymbol{S}$ by $(\boldsymbol{s}^1|\cdots|\boldsymbol{s}^m)$, we find the time evolution of $\boldsymbol{x}$,

$$\begin{aligned}\dot{\boldsymbol{x}} = S\dot{\boldsymbol{y}} &= \sum_i \boldsymbol{s}^i y_i({}^t\boldsymbol{s}^i A\boldsymbol{x} - {}^t\,\boldsymbol{x}A\boldsymbol{x}) \\ &= \sum_i y_i(\boldsymbol{s}^i - \boldsymbol{x})^t(\boldsymbol{s}^i - \boldsymbol{x})A\boldsymbol{x}. \end{aligned} \tag{3}$$

Time derivative $\dot{\boldsymbol{x}}$ is generally determined by the detailed structure of S.

A restriction of B by A, i.e., $rank(B) \le rank(A)$, effectively lowers the degrees of freedom of this dynamics. Moreover, the sets of points which are projected onto the equilibrium points of the original equation, eq. (1), by S are the equilibrium points of eq. (2) and they constitute an $(m-n)$-dimension hyperplane on the simplex S_{m-1}. Let $\tilde{\boldsymbol{q}} = \{\boldsymbol{y}|y_i = q_i(i = 1,\ldots,n), y_i = 0(i = n+1,\ldots,m)\}$, where $\boldsymbol{q}$ is an internal equilibrium point of the dynamics of eq. (1), $\boldsymbol{s}_i$ $(i = 1,\ldots,n)$ are pure strategies (i.e., $\boldsymbol{s}_i = \{\boldsymbol{s}|s_i = 1, s_j = 0(j \ne i)\}$ $(i = 1,\ldots,n)$), and $\tilde{\boldsymbol{q}}$ has $(n-1)$ stable directions on the simplex S_{n-1} and $(m-n)$ neutral directions on a hyperplane. Since the expected reward when the state is on the hyperplane is exactly the same as that of $\tilde{\boldsymbol{q}}$, mixed strategies can overtake the population by chance. However, the dynamical behaviors may be quite different along this hyperplane. As a main focus of this letter, we report that a three-pure-strategy game with two stable equilibriums (one at the corner and one internally in three-dimensional strategy space) can have an infinite number of periodic solutions consisting of mixed strategies when the internal equilibrium point is not ESS. This clearly shows that polymorphic solutions (mixed-strategy-involved) exhibit qualitatively different behaviors than solutions of pure strategies. Below, we introduce a payoff matrix under a certain condition and show that the above statement holds.

The game matrix of an evolutionary game system, with three-pure-strategies which have an internal equilibrium point, can be translated into the following form eq. (4). This translation does not change the topological characteristics of flows on the phase space.

$$A = \begin{pmatrix} 0 & d+a & d-a \\ d-b & 0 & d+b \\ d+c & d-c & 0 \end{pmatrix}. \tag{4}$$

It is easily shown that this game system has a unique internal equilibrium point $\boldsymbol{q}$ at $\frac{1}{3}(1,1,1)$. By means of a conventional linear stability analysis, we compute that a necessary and sufficient condition for making $\boldsymbol{q}$ stable is

$$d > 0 \quad \text{and} \quad ab + bc + ca > -d^2, \tag{5}$$

and a necessary and sufficient condition for making $\boldsymbol{q}$ an ESS is given by,

$$d > 0 \quad \text{and} \quad ab + bc + ca > (a^2 + b^2 + c^2) - 3d^2. \tag{6}$$

Thus a necessary and sufficient condition for making $\boldsymbol{q}$ stable but not an ESS is,

$$d > 0 \quad \text{and} \quad (a^2 + b^2 + c^2) - 3d^2 > ab + bc + ca > -d^2. \tag{7}$$

Without loss of generality, we assume $d = 1$. Under condition (7), we particularly focus on the following region where two saddles, which will be defined as $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, appear.

$$a > 1, \quad b > 1, \quad c < -1 \quad \text{and} \quad b + c > 0. \tag{8}$$

Because it is more complicated to prove the emergence of a new attracting set in general cases, here we assume condition (8). We have already confirmed that our argument here is valid in more general cases and the detailed proof will be reported elsewhere.

Under conditions (7) and (8), this game system has six equilibrium points (Fig. 1).

- $\boldsymbol{p}_1 = (1,0,0)$, $\boldsymbol{p}_2 = (0,1,0)$ and $\boldsymbol{p}_3 = (0,0,1)$ for the pure strategies.
- $\boldsymbol{r}_1 = (0, \frac{1+b}{2+b-c}, \frac{1-c}{2+b-c})$ on the line $\boldsymbol{p}_2$–$\boldsymbol{p}_3$ and $\boldsymbol{r}_2 = (\frac{1-a}{2-a+c}, 0, \frac{1+c}{2-a+c})$ on the line $\boldsymbol{p}_3$–$\boldsymbol{p}_1$ are equilibrium points.
- $\boldsymbol{q} = \frac{1}{3}(1,1,1)$ is the internal equilibrium point.

And we notice that attractors are $\boldsymbol{p}_1$ and $\boldsymbol{q}$, where $\boldsymbol{p}_1$ is ESS (i.e., $\forall\boldsymbol{x}$ $(\ne \boldsymbol{p}_1) \in S_2$, ${}^t\boldsymbol{p}_1 A\boldsymbol{p}_1 = 0 > {}^t\boldsymbol{x}A\boldsymbol{p}_1$) but $\boldsymbol{q}$ is not ESS. A basin boundary of $\boldsymbol{p}_1$ and $\boldsymbol{q}$ is given by a heteroclinic orbit from $\boldsymbol{p}_2$ to $\boldsymbol{r}_2$.

Now we introduce a mixed strategy $\boldsymbol{v} = (0, 1-k, k)$ using pure strategies 2 and 3. It is easy to extend this mixed strategy to use three pure strategies, but for simplicity we argue this two-pure-strategies case. Taking this strategy as the fourth corner, we embed S_2 onto S_3 as shown in Fig. 2. The above equilibrium points are

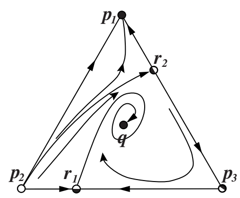

Fig. 1. Diagram of flow structure on the simplex S_2.There are two stable equilibrium points $\boldsymbol{p}_1$ and $\boldsymbol{q}$, and the basin boundary is given by the heteroclinic orbit from $\boldsymbol{p}_2$ to $\boldsymbol{r}_2$.

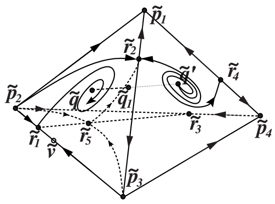

Fig. 2. Equilibrium points and flows on simplex S_3. Lines $\tilde{\boldsymbol{v}}$–$\tilde{\boldsymbol{p}}_4$, $\tilde{\boldsymbol{r}}_1$–$\tilde{\boldsymbol{r}}_3$ and $\tilde{\boldsymbol{q}}$–$\tilde{\boldsymbol{q}}'$ are parallel to each other. Points on lines $\tilde{\boldsymbol{q}}$–$\tilde{\boldsymbol{q}}'$ and $\tilde{\boldsymbol{r}}_1$–$\tilde{\boldsymbol{r}}_3$ are all equilibrium points. When the inequality $k^2(2+b-c) - 2k(1+a-c) + (2+a-b) > 0$ holds, $\tilde{\boldsymbol{r}}_4$ and heteroclinic orbit $\tilde{\boldsymbol{q}}' \to \tilde{\boldsymbol{r}}_4$ do not exist. In this case, $\tilde{\boldsymbol{p}}_4$ is unstable in the direction $\tilde{\boldsymbol{p}}_4$–$\tilde{\boldsymbol{p}}_1$.

accordingly embedded as follows:

- $\tilde{\boldsymbol p}_1 = (1,0,0,0)$, $\tilde{\boldsymbol p}_2 = (0,1,0,0)$, $\tilde{\boldsymbol p}_3 = (0,0,1,0)$
- $\tilde{\boldsymbol r}_1 = (0, \frac{1+b}{2+b-c}, \frac{1-c}{2+b-c}, 0)$, $\tilde{\boldsymbol r}_2 = (\frac{1-a}{2-a+c}, 0, \frac{1+c}{2-a+c}, 0)$
- $\tilde{\boldsymbol q} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ and $\tilde{\boldsymbol p}_4 = (0,0,0,1)$.

A set of points which is projected onto the equilibrium points by S of the pure strategy game system also consists of equilibrium points of this system. Here $\tilde{\boldsymbol q}$–$\tilde{\boldsymbol q}'$ and $\tilde{\boldsymbol r}_1$–$\tilde{\boldsymbol r}_3$ thus constitute the equilibrium points where $\tilde{\boldsymbol q}' = (\frac{1}{3}, 0, \frac{1-2k}{3(1-k)}, \frac{1}{3(1-k)})$ and $\tilde{\boldsymbol r}_3 = (0, 1 - \frac{1-c}{k(2+b-c)}, 0, \frac{1-c}{k(2+b-c)})$. They are projected onto $\boldsymbol q$ and $\boldsymbol r_1$ respectively by applying matrix S. We assume that parameter k satisfies the condition

$$\frac{1}{2} > k > \frac{1-c}{2+b-c}, \tag{9}$$

so that $\tilde{\boldsymbol r}_3$ is on $\tilde{\boldsymbol p}_2$–$\tilde{\boldsymbol p}_4$ and $\tilde{\boldsymbol q}'$ is on the plane including $\tilde{\boldsymbol p}_1$, $\tilde{\boldsymbol p}_3$ and $\tilde{\boldsymbol p}_4$. Under conditions (7), (8) and (9), $\tilde{\boldsymbol q}'$ is unstable on the plane $\tilde{\boldsymbol p}_1$–$\tilde{\boldsymbol p}_3$–$\tilde{\boldsymbol p}_4$. Since $\tilde{\boldsymbol q}$ is stable on plane $\tilde{\boldsymbol p}_1$–$\tilde{\boldsymbol p}_2$–$\tilde{\boldsymbol p}_3$, stability perpendicular to line $\tilde{\boldsymbol q}$–$\tilde{\boldsymbol q}'$ changes (see Fig. 2). Let us parameterize points on this line $\tilde{\boldsymbol q}$–$\tilde{\boldsymbol q}'$ by θ ($\tilde{\boldsymbol q}(\theta) = (1-\theta)\tilde{\boldsymbol q} + \theta\tilde{\boldsymbol q}' (0 \le \theta \le 1)$). From simple calculation, we found the point at which stability alternates at $\tilde{\boldsymbol q}_1$ $(= \tilde{\boldsymbol q}(\theta_1))$, where $\theta_1 = \frac{2}{(2+b-c)k}$. In particular, point $\tilde{\boldsymbol q}_1$ is a center whose eigenvalues have only imaginary parts.

There are several invariant curved surfaces, where no flows penetrating these curved surfaces exist. The following argument mainly focuses on the basin boundary between equilibrium points $\tilde{\boldsymbol q}$–$\tilde{\boldsymbol q}_1$ and $\tilde{\boldsymbol p}_1$. These correspond to the two equilibrium points of the original game system. The neighbors around $\tilde{\boldsymbol q}'$ are attracted to point $\tilde{\boldsymbol p}_1$. Since $\tilde{\boldsymbol q}$–$\tilde{\boldsymbol q}_1$ are stable fixed points, their neighbors are attracted to them. The basin boundary should be penetrated by the line connecting $\tilde{\boldsymbol q}$ and $\tilde{\boldsymbol q}'$. If no other attractors exist in this system, neighbors around $\tilde{\boldsymbol q}$–$\tilde{\boldsymbol q}_1$ are attracted to the points $\tilde{\boldsymbol q}$–$\tilde{\boldsymbol q}_1$ and those around $\tilde{\boldsymbol q}_1$–$\tilde{\boldsymbol q}'$ are attracted to $\tilde{\boldsymbol p}_1$.

Surprisingly, the situation is rather complicated and we have found a third attracting set. The existence of this attracting set is found numerically first and then confirmed analytically to some extent.

A precise boundary structure is obtained by the flow structure around line $\tilde{\boldsymbol q}$–$\tilde{\boldsymbol q}'$. We notice that a saddle point $\tilde{\boldsymbol r}_2$ has a two dimensional stable manifold. This manifold is a curved surface having two heteroclinic orbits on its boundary, which are $\tilde{\boldsymbol q}' \to \tilde{\boldsymbol r}_2$ on plane $\tilde{\boldsymbol p}_1$–$\tilde{\boldsymbol p}_3$–$\tilde{\boldsymbol p}_4$ and $\tilde{\boldsymbol p}_2 \to \tilde{\boldsymbol r}_2$ on plane $\tilde{\boldsymbol p}_1$–$\tilde{\boldsymbol p}_2$–$\tilde{\boldsymbol p}_3$ (see Fig. 2). We define Ω as this stable manifold of $\tilde{\boldsymbol r}_2$ below. Ω can be divided into two regions, one consisting of heteroclinic orbits from $\tilde{\boldsymbol p}_2$ and the other consisting of heteroclinic orbits from $\tilde{\boldsymbol q}(\theta)$ which are unstable. We label the former Ω_1 and the latter Ω_2 (see Fig. 3).

The boundary of Ω_1 and Ω_2 is a heteroclinic orbit to $\tilde{\boldsymbol r}_2$ from a point on line $\tilde{\boldsymbol r}_1$–$\tilde{\boldsymbol r}_3$. We label this point $\tilde{\boldsymbol r}_5$ below. There exists a heteroclinic orbit from $\tilde{\boldsymbol p}_2$ to $\tilde{\boldsymbol r}_5$ and also a heteroclinic orbit from $\tilde{\boldsymbol p}_3$ to $\tilde{\boldsymbol r}_5$. Thus Ω_2 has a heteroclinic cycle $\tilde{\boldsymbol p}_3 \to \tilde{\boldsymbol r}_5 \to \tilde{\boldsymbol r}_2$ on its boundary, and we call this heteroclinic cycle C.

Now we define a new critical point $\tilde{\boldsymbol q}_2 = \tilde{\boldsymbol q}(\theta_2)$ on line $\tilde{\boldsymbol q}$–$\tilde{\boldsymbol q}'$. There exists an orbit which starts from $\tilde{\boldsymbol q}_2$ and

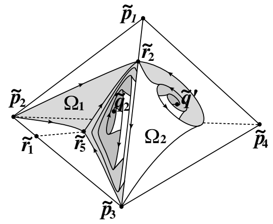

Fig. 3. Ω : A stable manifold of $\tilde{\boldsymbol r}_2$. Ω is divided into two regions, Ω_1 and Ω_2. Ω_1 is a set of heteroclinic orbits which starts from $\tilde{\boldsymbol p}_2$. Ω_2 is a set of heteroclinic orbits which starts from $\tilde{\boldsymbol q}(\theta)$. Their boundary is a heteroclinic orbit from $\tilde{\boldsymbol r}_5$ to $\tilde{\boldsymbol r}_2$.

composes the boundary of Ω_2. Judging from the facts that $\tilde{\boldsymbol q}_1$ is neutral, and that for θ close to θ_1, $\tilde{\boldsymbol q}(\theta < \theta_1)$ is stable and $\tilde{\boldsymbol q}(\theta > \theta_1)$ is unstable, we conclude that θ_2 is larger than θ_1. Indeed, $\tilde{\boldsymbol q}_2$ has an unstable manifold which has the heteroclinic cycle C on its boundary. We call this invariant curved surface α. Every point on α except $\tilde{\boldsymbol q}_2$ converges to C.

Due to the center manifold theorem, we have at least one center manifold through $\tilde{\boldsymbol q}_1$. Let us call this invariant surface β. Since $\tilde{\boldsymbol q}(\theta)$ $(\theta < \theta_1)$ has stable directions and $\tilde{\boldsymbol q}(\theta)$ $(\theta > \theta_1)$ has unstable directions, no point on β has $\tilde{\boldsymbol q}_1$ as α-limit or ω-limit set. Considering this and the fact that β is two-dimensional, every orbit on β has to be periodic. β has the heteroclinic cycle C as its boundary because C is the unique closed orbit including the boudary of the phase space. Thus, β is filled densely with periodic orbits.

We now have a region enclosed with α and β. Clearly, orbits in this region converge to neither $\tilde{\boldsymbol p}_1$ nor $\tilde{\boldsymbol q}$–$\tilde{\boldsymbol q}_1$ but to some attracting set. Since every orbit near $\tilde{\boldsymbol q}_1$ remains in the neighbor of $\tilde{\boldsymbol q}_1$ and since an unstable manifold of $\tilde{\boldsymbol q}(\theta)$ $(\theta > \theta_1)$ is two-dimensional, an unstable manifold of $\tilde{\boldsymbol q}(\theta)$ in the neighbor of $\tilde{\boldsymbol q}_1$ has a periodic orbit on β as its boundary. Since the unstable manifold of $\tilde{\boldsymbol q}_2$ has C as its boundary, and considering the continuity of flow structure, we see that every unstable manifold of $\tilde{\boldsymbol q}(\theta_1 < \theta < \theta_2)$ has a periodic orbit on β as its boundary.

As a result, we have found that center manifold β of $\tilde{\boldsymbol q}_1$ has a basin whose volume is positive and β becomes a new attracting set of the system.

We select the parameters under conditions (7)–(9) as $a = 5$, $b = 27$, $c = -3$ and $k = \frac{1}{4}$, where we found $\theta_1 = \frac{1}{4}$. Numerically, point $\tilde{\boldsymbol q}_2$ is found at $\theta_2 \simeq 0.3$.

We found that above θ_1 and under θ_2, there are regions which do not converge to $\tilde{\boldsymbol p}_1$ but converge to certain periodic solutions (Fig. 4). An example of an initial state which converges to the periodic orbit is depicted in Fig. 5.

We found that this attracting set has an uncountable infinite number of cycles on β standing in line around $\tilde{\boldsymbol q}_1$. It is bounded by the heteroclinic cycle C. This situation is depicted in Fig. 6.

As we mentioned earlier, if a three-strategy game has a non-ESS internal stable equilibrium point, we can always select a mixed strategy that generates an attracting set internally, consisting of an infinite number of periodic orbits. Discussions on the generic cases will be

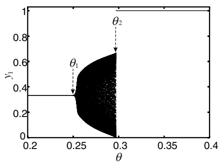

Fig. 4. θ–y_1: y_1 values of an orbit whose initial point is in the neighbor of $\tilde{\boldsymbol{q}}(\theta)$ are plotted against θ after a sufficiently long period of time. Orbits which started from the neighbors of $\tilde{\boldsymbol{q}}(\theta < \theta_1)$ are attracted to $\tilde{\boldsymbol{q}}(\theta < \theta_1)$ and orbits which started from the neighbors of $\tilde{\boldsymbol{q}}(\theta > \theta_2)$ are attracted to $\tilde{\boldsymbol{p}}_1$.

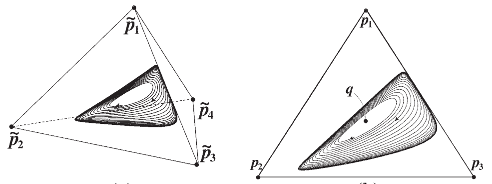

Fig. 5. An orbit converging to a new periodic motion. The initial state of this orbit is at $(\frac{5}{20}, \frac{6}{20}, \frac{6}{20}, \frac{3}{20})$. The orbit is plotted in S_3 in (a), and the orbit is projected by $S\boldsymbol{y}$ onto S_2 in (b).

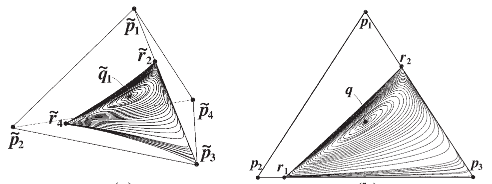

Fig. 6. The infinite number of periodic cycles constitutes an attracting set. They are projected onto S_2 by $S\boldsymbol{y}$ in (b).

reported elsewhere.

In this study, we have found a new attracting set by introducing a mixed strategy into game dynamics. A new strategy does not mean bringing a new action into the game system. It is no more than a stochastic combination of already existing actions. The game structure itself does not change, but the behavior changes nevertheless.

Attractors in the original system are all equilibrium points. However, a newly emerging attracting set consists of an infinite number of periodic cycles. This means that the effective degrees of freedom of the system have been increased by introducing a mixed strategy without changing the game structure itself.

The original system has an internal equilibrium point which is stable. The introduced strategy is a mixed strategy of two pure strategies; however, an equilibrium between the new strategy and the two pure strategies is set unstable. Such a situation occurs generally where the internal equilibrium point is stable but not an ESS (on the other hand, where the internal stable equilibrium point is an ESS, this situation never occurs). Therefore, the new attracting set found in this study is not a special case. Rather, it is a general characteristic of a game with mixed strategies. We have shown that a mixed strategy may expand the dimensionality of an attractor to form a new solution set, which was not discussed by Maynard Smith or in other related works.[2)]

From the point of view of evolutionary biology, introducing a mixed strategy may be considered as an appearance of a mutant whose character can be described merely by a recombination of the characters of pre-existing species. This situation corresponds to increasing the size of the interaction matrix without changing its rank size. When the system is at the internal equilibrium point, the mutant can invade and overtake the population by chance since the mutant has the same fitness (payoff). The appearance of a mutant often changes the structures of attractors; however, a completely new character is not required for a mutant to overtake the population.

We should discuss several other important situations: what happens if 1) more than one mixed strategy is introduced, 2) the internal attractors are not equilibrium points (*e.g.*, a limit cycle or quasi-periodic state) and 3) the stability of $\tilde{\boldsymbol{q}}$ and $\tilde{\boldsymbol{q}}'$ is reversed. These discussions will be reported elsewhere.

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